Combinatorial Game Theory

Kyle Flanagan and Hayden Ringer

Math Circle

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- Brussel Sprouts is a 2-player game, where the players alternate turns.
  - The game starts with some predetermined number of crosses.
  - A player makes a valid move by connecting two separate free ends of the crosses on the board with a line, and adding a new free end on both sides of the line.
  - The last player to be able to make a move wins.
- Example of the first two moves made during a game:

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  - The game must finish in finite moves.
  - There are no ties and no scoring.

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  - There are no ties and no scoring.
- We say that the game is in normal play if the the last player to move wins. If the last player to move loses, then the game is in misère play.

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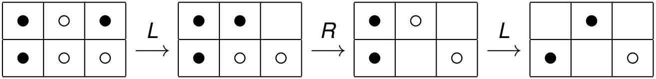
- Brussel Sprouts
- Nim
- Penny Placer
- Chomp
- Clobber
    - Players play on an $n \times m$ board, where each square is empty or is occupied by a black or a white stone.
    - Left must move a black stone onto a vertically or horizontally adjacent white stone and remove the white stone. Similarly for Right moving white stones.
    - Left typically moves first.
    - Normal play (i.e., last player to move wins)

Clobber Game Example

- Here is a simple game of Clobber with a 2×3 board



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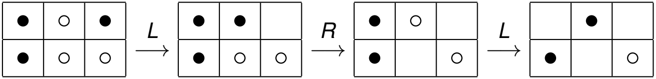

- Notice that Left wins here because Right cannot make any legal moves.

The Fundamental Theorem

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Theorem (The Fundamental Theorem of Combinatorial Games)

Consider a combinatorial game between Left and Right with Left moving first. Then (under perfect play) either Left can force a win moving first, or Right can force a win moving second, but not both.

- This means that at any point during a combinatorial game, if both players had perfect information about the game and make no mistakes, either the current player can guarantee they win the game or they cannot win the game no matter what moves they make.

Outcome Classes

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- A game where Left and Right have the same moves or options is called an impartial game, which we will now solely focus on.
- The previous theorem implies that every position of a game belongs to exactly one of the following outcome classes.
  - $\mathcal{N}$ - $\mathcal{N}$ext player to play (whether Left or Right) can force a win
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- We will say a position is an $\mathcal{N}$-position or a $\mathcal{P}$-position, etc.
- A player wins if they move the game to a $\mathcal{P}$-position.
- A primary goal of combinatorial game theory is to know the $\mathcal{P}$-positions.

2-Pile Nim

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- Rules of Nim(m, n)
  - Players play on 2 piles of counters where the first pile has m counters and the second pile has n counters.
  - Players remove a nonzero number of counters, up to the total number of counters, from exactly one nonempty pile.
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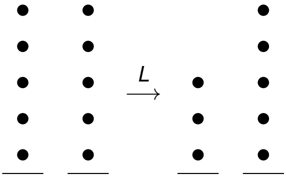
- Here is an example first move for Nim($5, 5$):



Symmetry in 2-Pile Nim

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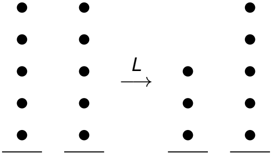
- Consider the same starting move to Nim$(5, 5)$



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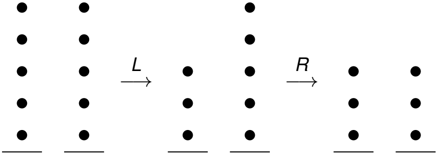

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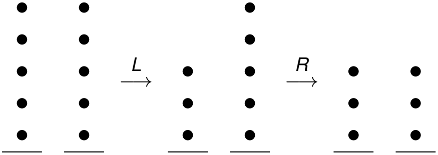

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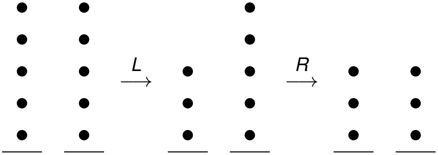

- We can now observe that a position in 2-Pile Nim is a $\mathcal{P}$-position if and only if the two piles have the same number of counters.
- This kind of symmetry argument will not always work in a game of Nim with more than two piles.

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  - The first player places their first penny in the center of the board. Then for each subsequent turn, the first player places their penny on the board 180° from the penny the second player just placed.

Modified Nim

Modified Nim

- Rules of MNim(m, n)
  - Same rules as Nim(m, n) but players have an additional option to skip their own turn.
  - Each player can only skip their turn once per game, which is synonymous with taking no counters from either pile.
  - A player cannot skip if the on the previous turn, the other player skipped.

Chomp

- Chomp(m, n)
  - Players play on an $m \times n$ board of squares where the bottom left corner is poisonous
  - Players must remove a square and all other squares above it and to the right of it.
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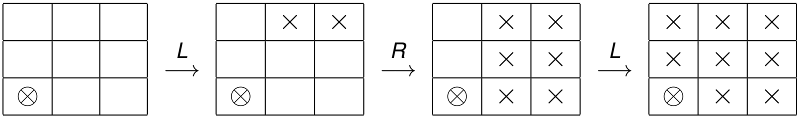
- Here is an example game of Chomp$(3, 3)$ where Left wins



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Proof.

Suppose the first player chomps only the upper-right square. If this is a $\mathcal{P}$-position, then the starting position was $\mathcal{N}$. If not, then player two can win by chomping all squares above and to the right of some square S. However, the first player had this move available to play during their first turn. Therefore, the original position was actually an $\mathcal{N}$-position. ∎

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- This type of argument is called strategy stealing, and tells you nothing about the strategy of the game.

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Let m denote the numbers of moves played throughout the game, and c the number of initial crosses. We will denote by v, e, f the number of of vertices, edges, and faces of the planar graph obtained at the end of the game.

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Proof.

- Notice that $e = 2m$ because at each move the player adds 2 edges.
- Notice that $v = c + m$ because at each move the player adds one vertex, and the game starts with c vertices.
- Notice that $f = 4c$ because there is exactly one free end in each face at the end of the game, and the number of free ends never changes during the game.

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Proof.

The Euler characteristic for planar graphs is $f - e + v = 2$, so

$$\begin{aligned} 2 = f - e + v &= 4c - 2m + (c + m) \\ &= 5c - m. \end{aligned}$$

Hence, $m = 5c - 2$ and so the game was predecided based on the number of crosses. ∎

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Therefore, Brussel Sprouts on c crosses is a $\mathcal{P}$-position if and only if c is even.